Physics of Fluids (WBPH042-05)

8th of April 2025, 18:15-20:15

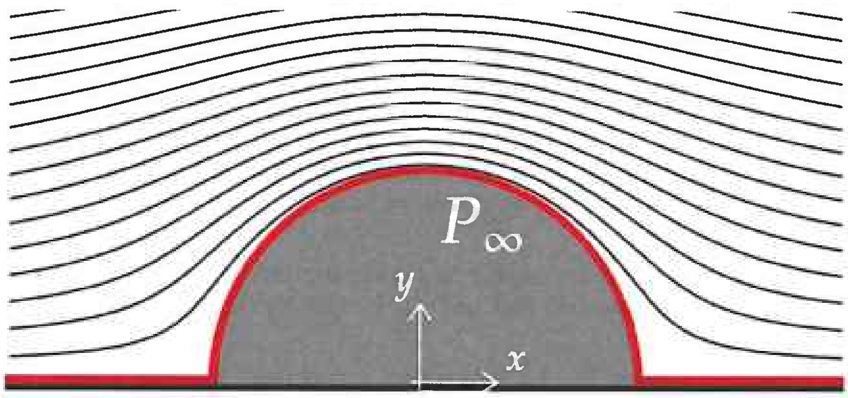
This is a closed-book exam; only simple pocket calculators are allowed. There are 3 questions in total.

When, for some reason, you are unable to answer part of questions (a, b, etc.), make a realistic assumption and use this for the rest of the question.

Write your answer to each of the 3 questions **on a separate answer sheet**. Please write your name and student number *on each answer sheet* that you hand in.

GOOD LUCK!

1 2D flow around a Quonset hut [18 Points]



The fluid velocity around a semicircular quonset hut of radius a is given by

$$\begin{pmatrix} u_x \\ u_y \end{pmatrix} = U_\infty \begin{pmatrix} 1 + \frac{a^2(y^2-x^2)}{(x^2+y^2)^2} \\ -\frac{2a^2xy}{(x^2+y^2)^2} \end{pmatrix}, \tag{1}$$

where U_∞ is the velocity far upstream at pressure P_∞. Assume that the pressure inside the hut is also P_∞.

(a) [6 Points] Show that the fluid velocity can be written in polar coordinates (r, θ) as

$$\begin{pmatrix} u_r \\ u_\theta \end{pmatrix} = U_\infty \begin{pmatrix} (1 - \frac{a^2}{r^2}) \cos\theta \\ -(1 + \frac{a^2}{r^2}) \sin\theta \end{pmatrix}. \tag{2}$$

(b) [4 Points] Show that the curve

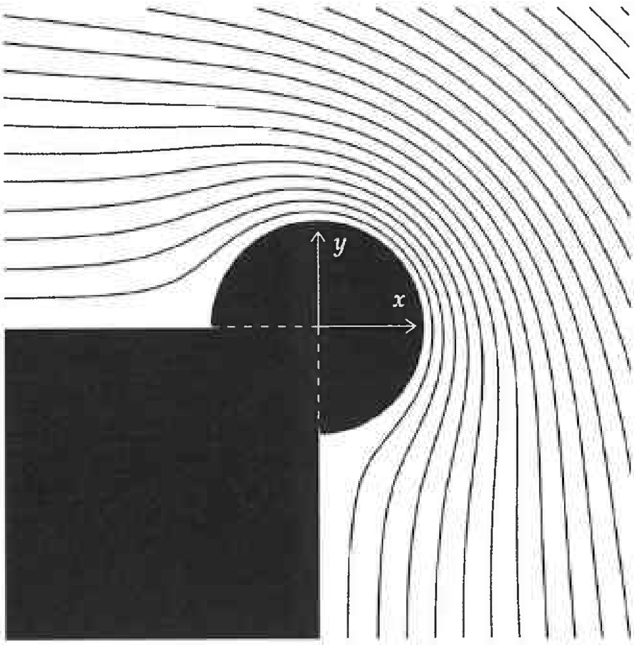
$$y = \begin{cases} \sqrt{a^2 - x^2} & \text{if} \quad x \in [-a, a], \\ 0 & \text{in any other case,} \end{cases} \tag{3}$$

is composed of streamlines (that is, it is tangent to the fluid flow).

(c) [4 Points] Formulate a scaling law for the vertical force per unit out-of-plane thickness, f_y, experienced by the house due to the fluid as a function of U_∞, a, and the fluid density ρ. Assume that the flow occurs at high enough Reynold's number to ignore viscosity.

(d) [4 Points] Should the proportionality constant in the scaling law be positive or negative? Explain your reasoning [Hint: Think of the changes in pressure].

2 Ideal flow around a corner with a bump [22 Points]



Consider a two-dimensional, steady, incompressible (density ρ), inviscid, and irrotational flow around a square corner with a circular bump. The stream function $\psi(r,\theta)$ is given by:

$$\psi(r,\theta) = A\left(r^{2/3} - r^{-2/3}\right)\sin\left(\frac{2}{3}\theta + \frac{\pi}{3}\right), \tag{4}$$

where A is a non-zero constant, r is the distance from the origin measured in cm, and θ is restricted to be between $(-\pi, \pi]$. In the figure, the black lines represent the streamlines.

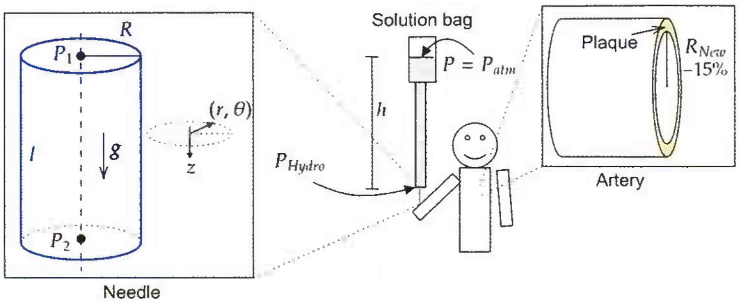
(a) [6 Points] Calculate the velocity potential $\phi(r,\theta)$ and sketch its equipotential lines.

(b) [5 Points] Show that the contour line $\psi = 0$ defines the boundary of the square with the circular bump. Determine the radius of the circular bump.

(c) [5 Points] Find the stagnation points of the flow.

(d) [6 Points] Use the steady Bernoulli equation to determine the pressure along the $\psi = 0$ boundary. Assume that the pressure at infinity is zero.

3 Needles and arteries: Poiseuille flow [22 Points]



Consider a cylindrical tube of radius R as a model for a needle in which an incompressible fluid of viscosity μ and density ρ is injected by a constant pressure gradient $-G$ and gravity g, the latter acting along the cylinder. Assume a fully developed, steady, axisymmetric laminar flow parallel to the cylinder axis. [Hint: The Navier-Stokes equations for these conditions simplify to $\nabla \cdot \boldsymbol{u} = 0$ and $\mu\nabla^2\boldsymbol{u} - \nabla P + \rho\boldsymbol{g} = \boldsymbol{0}$]

(a) [6 Points] By assuming the non-slip boundary condition, show that the velocity field satisfies

$$\boldsymbol{u} = u_z(r)\hat{\boldsymbol{e}}_z, \qquad u_z(r) = \frac{G + \rho g}{4\mu}(R^2 - r^2). \tag{5}$$

(b) [4 Points] Show that the volume flow rate through the tube, Q, is given by

$$Q = \frac{\pi(G + \rho g)R^4}{8\mu}. \tag{6}$$

(c) [4 Points] A patients blood pressure reads 8.00 mm of Hg above atmospheric pressure. What pressure must be supplied at the top of the needle for an intravenous system if a salted solution of 0.120 cm^3/s must be supplied through a needle of length 2.50 cm and radius of 0.150 mm? [Hint: Assume that the pressure drop is linear along the length of the needle ($G = (P_1 - P_2)/l$), and that the density and viscosity of the salted solution are the same water. $\mu_{H_2O} = 1.002\text{mPa}\cdot\text{s}$. 1mm Hg = 133.32239 Pa. P_{atm}= 760 mm Hg].

(d) [4 Points] At what height, h, should the solution bag be held if the pressure P_1 must be supplied by the hydrostatic pressure? For this problem, assume that the free boundary of the bag is at atmospheric pressure.

(e) [4 Points] If we ignore the effect of gravity ($g = 0$), Eq. 6 can also be used to model the flow of blood through the arteries. Suppose that the radius of an artery is reduced by 15% due to plaque deposition caused by high cholesterol levels. By what percentage is the mass flow rate reduced? How much should the pressure gradient increase to supply the same mass flow rate as in a healthy person? [Note that the pressure increase would have to be supplied by the heart, causing severe strain to the circulatory system].

Formulae

$$\begin{aligned} \boldsymbol{A} &= A_x \hat{\boldsymbol{e}}_x + A_y \hat{\boldsymbol{e}}_y + A_z \hat{\boldsymbol{e}}_z && (7) \\ &= A_r \hat{\boldsymbol{e}}_r + A_\theta \hat{\boldsymbol{e}}_\theta + A_z \hat{\boldsymbol{e}}_z && (8) \end{aligned}$$

$$\begin{cases} \hat{\boldsymbol{e}}_r = +\cos\theta\, \hat{\boldsymbol{e}}_x + \sin\theta\, \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_\theta = -\sin\theta\, \hat{\boldsymbol{e}}_x + \cos\theta\, \hat{\boldsymbol{e}}_y \\ \hat{\boldsymbol{e}}_z = \hat{\boldsymbol{e}}_z \end{cases} \tag{9}$$

$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2} \tag{10}$$

$$\begin{aligned} \nabla^2 \boldsymbol{A} &= \left(\nabla^2 A_r - \frac{A_r}{r} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\hat{\boldsymbol{e}}_r && (11) \\ &+ \left(\nabla^2 A_\theta - \frac{A_\theta}{r^2} + \frac{2}{r^2}\frac{\partial A_\theta}{\partial \theta}\right)\hat{\boldsymbol{e}}_\theta && (12) \\ &+ \left(\nabla^2 A_z\right)\hat{\boldsymbol{e}}_z && (13) \end{aligned}$$

$$\begin{cases} u_r &= \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \\ u_\theta &= \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} \end{cases} \tag{14}$$